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A reformulation of the 3x+1 function revealing some structural regularity

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1 Introduction

This paper reformulates the $3x+1$ problem using term rewriting rules on abstract structures representing the binary encoding of numbers. The abstract structures and the rules have been used to derive a binary relation $\ddot{\prec}$ over odd numbers such that $\ddot{T}(x) \ddot{\prec} x$, for all x in odd numbers. To prove the Conjecture it is sufficient to prove that the transitive closure of this binary relation is antisymmetric, is irreflexive (except for 1), and has 1 as the smallest element.

The $3x+1$ problem concerns the iteration of the function $T : N^+ \rightarrow N^+$, ore equivalently of function $\ddot{T} : 2N+1 \rightarrow 2N+1$, defined as: where N is the set of all natural numbers, N^+ is the set of positive natural

$$T(x) = \begin{cases} x/2 & x \text{ is even} \\ \frac{3x+1}{2} & x \text{ is odd} \end{cases} \qquad (1) \qquad\qquad \ddot{T}(x) = \frac{3x+1}{2^j} \qquad (2)$$

numbers, $3x+1 \bmod 2^j = 0$ and $3x+1 \bmod 2^j \neq 0$. The $3x+1$ Conjecture asserts that repeated iteration of the function T (or equivalently $\ddot{T}$) on any positive integer x eventually produces 1 [1, 3].

The recursive application of the $3x+1$ function is a good example of a program that is known to halt for all numbers that it has been tried on. However, there does not yet exist a proof that such a program would indeed halt for all numbers. In a recent paper Margenstern develops a $3x+1$ *machine*, a machine based on the $3x+1$ function, to model computations and to develop boundaries between decidable and undecidable problems [2]. Understanding of the $3x+1$ Conjecture could thereby provide insight into understanding other problems that may be modelled using it.

2 Reformulation

The first column of Figure 1 defines the syntax domains B, B^+, B^1,andB^0, representing binary sequences representing the set of all numbers, the set of all positive numbers, the set of all even numbers, and the set of all odd numbers. The symbol "." in these figures is a *token*. (It may loosely be interpreted as the concatenation operator.) The symbols $k0$ and $j1$, as defined in the figure, represent an even number and an odd number, respectively.

Some of the structures created by this grammar are: 0, 0.01, 0.10, $0.01.1$, $0.10.10$, $0.01.01$. They represent the numbers 0, 1, 2, 3, 4, and 5, respectively. All terms must begin with a 0. There may, however, be redundant zeros as well, such as 0.0, $0.0.0$, $0.0.0.01$, $0.0.10$. We assume that terms with redudant zeros are mapped to their equivalent terms by removing the leading zeros.

The second column of Figure 1 presents a function, $\ddot{T'} : B^1 \rightarrow B^1$. The functions it uses are presented in the first and third columns of Figure 2. It can be shown that $\ddot{T'}$ is isomorphic to $\ddot{T}$ by deriving it from the equations of T.

$b \in B$	$\ddot{T}' : B^1 \rightarrow B^1$	$\ddot{\prec} : B^1 \times B^1$	
$k0, k'0, k''0 \in B^0$	$\ddot{T}'(0.01) = 0.01$	$0.01 \ddot{\prec} 0.01$	
$j1, j'1, j''1 \in B^1$			
$B^+ = B - \{0\}$	$\ddot{T}'(j1.1.1) = f_1(j1).1$	$j'1.1 \ddot{\prec} j1.1.1$	if $j'1 \prec j1$
$b ::= 0 \mid k0 \mid j1$	$\ddot{T}'(j1.01.1) = f_0(j1).01$	$k0.01 \ddot{\prec} j1.01.1$	if $k0 \prec j1$
$k0 ::= 10 \mid k'0.0 \mid k''0/10 \mid j1.10$	$\ddot{T}'(k0.01.1) = h_1(k0).01$	$j'1.01 \ddot{\prec} k0.01.1$	if $j'1 \prec k0$
$j1 ::= 1 \mid j'1.1 \mid j''1.01 \mid k0.01$			
	$\ddot{T}'(j1.01) = \ddot{T}'(j1)$	$j'1 \ddot{\prec} j1.01$	if $j'1 \prec j1$
	$\ddot{T}'(k0.0.01) = h(k0).01$	$b.01 \ddot{\prec} k0.0.01$	if $b \prec k0$
	$\ddot{T}'(k0.10.01) = h_1(k0).1.1$	$j1.1.1 \ddot{\prec} k0.10.01$	if $j1 \prec k0$
	$\ddot{T}'(j1.10.01) = f(j1).01.1$	$b.01.1 \ddot{\prec} j1.10.01$	if $b \prec j1$

Figure 1 A term rewriting reformulation of function T and the constraints for $\prec$ derived from each equation to satisfy the condition $\ddot{T}'(x) \ddot{\prec} x$, for all x in B^1.

Observe the symmetry between the equations in columns 1 and 3 of Figure 2. Equations in one column can be derived from the equation of the same row in the other column by replacing all occurrences of f by h, h by f, 1 by 0, 0 by 1, k by j, and j by k. The exceptions are equations that deal with the base case.

Margenstern's $3x+1$ machine is based on a finite state machine that computes the $3x+1$ function with a stream of binary numbers [2]. The forumlation presented here was developed independently of it. The syntax domain and rewrite rules presented here provide a richer representation to apply structural induction than Margenstern's state machine.

3 Derivation of a binary relation

The third column of Figure 1 gives constraints for the relation $\ddot{\prec} : B_1 \times B_1$. Each consraint for $\ddot{\prec}$ is paired with the equation in the same row. A constraint is *derived* from its corresponding equation. It describes the property that should be satisfied by the relation $\ddot{\prec}$ in order for the corresponding equation to ensure that $\ddot{T}'(x) \ddot{\prec} x$.

Consider, the base case $\ddot{T}'(0.01) = 0.01$. This equation will ensure that $\ddot{T}'(0.01) \ddot{\prec} 0.01$ if $0.01 \ddot{\prec} 0.01$.

The next equation $\ddot{T}'(j1.1.1) = f_1(j1).1$, may be rewritten as $\ddot{T}'(j1.1.1) = j'1.1$, where $j'1 = f_1(j1)$. To ensure that $\ddot{T}'(j1.1.1) \ddot{\prec} j1.1.1$, it is imperative that $j'1.1 \ddot{\prec} j1.1.1$ under the condition that $f_1(j1) \prec j1$, i.e., $j'1 \prec j1$. The other constraints are derived similarly.

Columns 2 and 4 of Figure 2 contains the constraints for $\prec$ derived from their corresponding equations..

From construction of the constraints it can be shown using structural induction that $\ddot{T}'(x) \ddot{\prec} x$, for all $x \in B_1$ and that for every function $\xi \in \{f, f_0, f_1, h, h_0, h_1\}$, $\xi(x) \prec x$, for every element x in their respective domains.

Definition: A binary relation $\prec : D \times D$ is P-reflexive if for all $x, y \in D$, $x \prec y$ and $y \prec x$ iff $x = y$ and $x \in P$.

Now consider the relation $\ddot{\prec}^+ : B^1 \times B^1$, the transitive closure of $\ddot{\prec}$. To show that the $3x+1$ Conjecture is valid it is sufficient to show that (a) $\ddot{\prec}^+$ is antisymmetric, (b) $\ddot{\prec}^+$ is $\{1\}$–reflexive, and that (c) 0.01 (the representation for the numeric value 1) is the smallest element in B^1, i.e., for all $x \in B^1$, $0.01 \ddot{\prec}^+ x$.

On the other hand, invalidation of any of the above condition does not invalidate the Conjecture. Thus, proving that the relation $\ddot{\prec}^+$ satisfies the above conditions is sufficient to prove the conjecture. If the constraints

$f : B^1 \to B$	$\prec : B \times B$		$h : B^0 \to B$		
			$h(0) = 0$	$0 \prec 0$	
$f(j1.1) = f_1(j1)$	$j'1 \prec j1.1$	if $j'1 \prec j1$	$h(k0.0) = h_0(k0)$	$k'0 \prec k0.0$	if $k'0 \prec k0$
$f(j1.01) = f_0(j1).0$	$k0.0 \prec j1.01$	if $k0 \prec j1$	$h(k0.10) = h_1(k0).1$	$j1.1 \prec k0.10$	if $j1 \prec k0$
$f(k0.01) = h(k0).10$	$b.10 \prec k0.01$	if $b \prec k0$	$h(j1.10) = f(j1).01$	$b.01 \prec j1.10$	if $b \prec j1$
$f_0 : B^1 \to B^0$			$h_1 : B^0 \to B^1$		
			$h_1(0) = 0.01$	$0.01 \prec 0$	
$f_0(j1.1) = f(j1).10$	$b.10 \prec j1.1$	if $b \prec j1$	$h_1(k0.0) = h(k0).10$	$b.01 \prec k0.0$	if $b \prec k0$
$f_0(j1.01) = f_0(j1).0.0$	$k0.0.0 \prec j1.01$	if $k0 \prec j1$	$h_1(k0.10) = h_1(k0).1.1$	$j1.1.1 \prec k0.10$	if $j1 \prec k0$
$f_0(k0.01) = h(k0).10.0$	$b.10.0 \prec k0.01$	if $b \prec k0$	$h_1(j1.10) = f(j1).01.1$	$b.01.1 \prec j1.10$	if $b \prec j1$
$f_1 : B^1 \to B^1$			$h_0 : B^0 \to B^0$		
			$h_0(0) = 0$	$0 \prec 0$	
$f_1(j1.1) = f_1(j1).1$	$j'1.1 \prec j1.1$	if $j'1 \prec j1$	$h_0(k0.0) = h_0(k0).0$	$k'0.0 \prec k0.0$	if $k'0 \prec k0$
$f_1(j1.01) = f_0(j1).01$	$k0.01 \prec j1.01$	if $k0 \prec j1$	$h_0(k0.10) = h_1(k0).10$	$j1.10 \prec k0.10$	if $j1 \prec k0$
$f_1(k0.01) = h_1(k0).01$	$j1.01 \prec k0.01$	if $j1 \prec k0$	$h_0(j1.10) = f_0(j1).10$	$k0.10 \prec j1.10$	if $k0 \prec j1$

Figure 2 Equations for functions used in defining $\ddot{T}'$ and constraints for relation $\prec$ derived from the equations.

for $\ddot{\prec}$ presented do not satisfy the conditions, it is likely that constraints derived by expanding (or unfolding) the equations may lead to the desired set.

4 Conclusions

I hope that the new formulation of the $3x + 1$ problem presented in this paper may provide insight into an ordering over which $T(x) \prec x$.

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