A reformulation of the 3x+1 function revealing some structural regularity

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Abstract

This paper reformulates the 3x + 1 problem using term rewriting rules on abstract structures representing the binary encoding of numbers. The abstract structures and the rules reveal some regularities in the function that are not obvious in the usual numeric formulation.

1 Introduction

The 3x + 1 problem concerns the iteration of the function $T : N^+ \rightarrow N^+$ defined as:

$$T(x) = \begin{cases} 
\frac{x}{2} & \text{x is even} \\
\frac{3x+1}{2} & \text{x is odd}
\end{cases}$$

or equivalently as:

$$T(2x) = x$$
$$T(2x + 1) = 3x + 2 = (2x + 1) + x + 1$$

where $N^+$ is the set of positive natural numbers. The 3x + 1 Conjecture asserts that repeated iteration of the function $T$ on any positive integer $x$ eventually produces 1. The conjecture according to Lagarias was first formulated by Lothar Collatz in early 1930s [2]. The Conjecture is important in that insights into the iterative properties of function $T$ may lead to insights into other Diophantine equations, a class of functions to which $T$ belongs. Further details about known results on the problem and its generalization may be found in survey papers [2, 4].

In this paper I present a reformulation of the 3x + 1 function as term rewriting rules on a structure representing the binary encoding of a number. The term rewriting rules, being operations on structures, are written without the “+” and “/” operation. The reformulation is done in two steps. First, I restate the problem as operation on sequences of binary digits. Next, I derive a collection of equalities which collectively define the computation of 3x + 1 on binary sequences. The new formulation is arrived at by replacing the original function by the equalities.
2 Reformulation

Let $B$ be the set of sequences given by the regular expression $\{1\{0 \mid 1\}^* \mid 0\}$. Since elements of $B$ are binary encoding of natural numbers, the set $B$ is isomorphic to $\mathbb{N}$, the set of natural numbers. Let $B^+ = B - \{0\}$, be the set of binary sequences, excluding the sequence $0$. Thus, $B^+$ is isomorphic to $\mathbb{N}^+$. 

**Notation:** If $r$ is a binary sequence then $r0$ (similarly, $r1$) is the sequence created by concatenating to the right of $r$ the element 0 (similarly, 1).

Let $\varepsilon$, the null sequence, be the identity of concatenation. That is $r\varepsilon = \varepsilon r = r$. For any set $S$ let $S_\varepsilon$ be the set $S \cup \{\varepsilon\}$. Let $\oplus : B_\varepsilon \times B_\varepsilon \rightarrow B$ be the addition operator. It treats the null sequence $\varepsilon$ as the value 0.

Now consider the function $T' : B^+ \rightarrow B^+$.

$$
T'(k0) = k
$$

$$
T'(j1) = j1 \oplus j \oplus 1
$$

where in the above equations and the rest of the paper $j \in B^+_\varepsilon$ and $k \in B^+$.

**Lemma:** Functions $T$ and $T'$ are isomorphic.

**Proof:** From construction.

Let $B^1 \subset B^+$ be the set of positive binary sequences terminating with 1 and $B^0 \subset B^+$ be the set of positive binary sequences terminating with 0. The sets $B^0$ and $B^1$ represent the set of even and odd numbers, respectively. $B^0$ and $B^1$ partition $B^+$.

We now introduce two functions $f : B^1 \rightarrow B^+$ and $h : B^0 \rightarrow B^+$.

$$
f(j1) = j1 \oplus j \oplus 1
$$

(4)

$$
h(k0) = k0 \oplus k
$$

(5)

The function $f$ essentially computes $(3x + 1)/2$, when $x$ is an odd, positive number and the function $h$ computes $3x/2$, when $x$ is an even, positive number.

**Lemma:** The following equalities hold.

\[
\begin{align*}
h(10) &= 11 & f(1) &= 10 \\
h(k0.0) &= h(k0)0 \\
h(k0.10) &= h(k0).11 \\
h(j1.01) &= f(j1)01
\end{align*}
\]

(6a) \quad (6b) \quad (6c) \quad (6d)

\[
\begin{align*}
h(10) &= 11 & f(1) &= 10 \\
h(k0.0) &= h(k0)0 \\
h(k0.10) &= h(k0).11 \\
h(j1.01) &= f(j1)01
\end{align*}
\]

(7a) \quad (7b) \quad (7c) \quad (7d)

where “.” is used just for punctuation.

**Proof:** Follows from the definitions of functions $f$ (equation 4), function $h$ (equation 5), binary addition operator $\oplus$, and concatenation.

Now I will show that equations 6a-d define the function $h$ and equations 7a-d define the function $f$. Since the said equations are also in terms of $f$ and $h$, to reduce confusion I will use the symbols $\hat{f}$ and $\hat{h}$ to represent the functions defined in equation 4 and 5. The assertion to be proven follows:
Theorem: Functions $\hat{h} : B^0 \to B^+$ and $\hat{f} : B^1 \to B^+$ are unique solutions of the recursive equations 6a-d and 7a-d, respectively.

Proof: In the following discussion.

There are two parts to the proof. First, that the equations 6a-d and 7a-d satisfy the requirements for structural induction. Second, that the functions $\hat{h}$ and $\hat{f}$ are unique solutions of the respective equations.

To satisfy the requirements for structural induction I will develop the syntax domain over which the equations are defined.

In equations 6a-d consider the terms $k0.0$, $k0.10$, $j1.10$. Any element in $B^0$ matches one and only one of these terms. Thus, these terms partition $B^0$. Similarly, the terms $1$, $j1.1$, $j1.01$, $k0.01$ of equations 7a-d partition the set $B^1$. Since the term $j1$ creates elements from $B^1$ and the term $k0$ creates elements from $B^0$, the sets $B^0$ and $B^1$ may be described by the following abstract syntax domains:

$$
\begin{align*}
  k0, k'0, k''0 & \in B^0 \\
  j1, j'1, j''1 & \in B^1 \\
  k0 & ::= 10 \mid k'0.0 \mid k''0.10 \mid j1.10 \\
  j1 & ::= 1 \mid j'1.1 \mid j''1.01 \mid k0.01
\end{align*}
$$

Each equation of 6a-d and 7a-d constrains one structure in these abstract syntax domains and the right-hand side of each equation is a function of the subcomponent of the structure in the left-hand side. Thus, the equations satisfy the requirements for applying structural induction. It also follows that these equations have a unique solution [3, Theorem 3.13].

That $\hat{h}$ and $\hat{f}$ are solutions of equations 6a-d and 7a-d, respectively, can be determined by structural induction as follows: For each equation, replace $\hat{h}$ for $h$ and $\hat{f}$ for $f$, and simplify to show that the two sides are equal.

We can now rewrite the function $T'$ as follows:

$$
T'(k0) = k \\
T'(j1) = f(j1)
$$

where $h$ and $f$ are defined by equations 6a-d and 7a-d, respectively.

The above describes the function $T'$ entirely using term rewriting rules, with just one caveat. Consider the term $f(j1)1$ in equation 7b. Since it terminates with 1, it clearly belongs to the domain $B^1$. The question arises what is its structure. Is its structure described by $B^{11}$, $B^101$, or $B^001$? Similar questions may be raised about the terms in the RHS of the other equations.

Since one cannot associate a structure in the syntax domain $B^0$ or $B^1$ to the RHS of the equations 6a-d and 7a-d, the range of the functions $T'$, $f$, and $h$ may best be described by the syntax domain $B^+$, as defined below:

$$
\begin{align*}
  b, b', b'' & \in B^+ \\
  b & ::= 1 \mid b'0 \mid b''1
\end{align*}
$$

While the concrete syntax of $B^+$ is the union of the concrete syntax of $B^0$ and $B^1$, the issue at hand is that the range of functions $T'$, $f$, and $h$ is not the union of the abstract syntax domains $B^0$ and $B^1$. So even though the formulation of $T'$ is isomorphic to $T$, it is not very conducive for investigating its iterative properties using structural induction.
I now define \( \tilde{T} : 2N + 1 \rightarrow 2N + 1 \) an alternate form of the function \( T \), taken from Andaloro [1], and then derive a term rewriting formulation for this function.

\[
T(x) = \frac{3x + 1}{2^j}
\]

(11)

where \( 3x + 1 \mod 2^j = 0 \) and \( 3x + 1 \mod 2^j \neq 0 \). Function \( \tilde{T} \), though not equivalent to function \( T \), has the same characteristic so far as the \( 3x + 1 \) Conjecture is concerned.

Function \( \tilde{T}' : B^1 \rightarrow B^+ \) defined by the equations below is derived from equations 7 defining \( f \).

\[
\begin{align*}
\tilde{T}'(1) &= 1 \\
\tilde{T}'(j1.1) &= f(j1)^1 \\
\tilde{T}'(j1.01) &= \tilde{T}'(j1) \\
\tilde{T}'(k0.101) &= h(k0)^1
\end{align*}
\]

(12)

The above equations were derived using the following steps: (a) The symbol \( f \) in the LHS of the equations 7 was replaced by the symbol \( \tilde{T}' \). (b) The term \( f(j1)^00 \) in the RHS of the third equation was replaced by \( \tilde{T}'(j1)^000 \). (c) The trailing 0s from all the terms in the RHS were removed (resulting into division by \( 2^2 \)).

**Lemma**: Functions \( \tilde{T} \) and \( \tilde{T}' \) are isomorphic.

**Proof**: From construction.

The terms in the RHS of all the equations for \( \tilde{T}' \) (12) create binary sequences that end with 1, thus generating odd numbers. However, it is still not apparent which of the four structures of \( B^1 \) they create. I now specialize the functions \( f \) and \( h \) to create a collection of functions as follows:

\[
\begin{align*}
f_0(j1) &= f(j1)^0 \\
f_1(j1) &= f(j1)^1 \\
f_{10}(j1) &= f(j1)^{10} \\
f_{01}(j1) &= f(j1)^{01}
\end{align*}
\]

\[
\begin{align*}
h_1(k0) &= h(k0)^1 \\
h_0(k0) &= h(k0)^0 \\
h_{01}(k0) &= h(k0)^{01} \\
h_{10}(k0) &= h(k0)^{10}
\end{align*}
\]

The complete reformulation of function \( \tilde{T}' \) along with the specialized functions is given in Figure 1. The range of all of these functions is known to be either \( B^0 \) or \( B^1 \). Henceforth, we can consider \( B^+ \) to be a union of \( B^0 \) and \( B^1 \).

I’ll now study the structure of the domains \( B^0 \) and \( B^1 \). Does the domains have a least element? Is 1 the smallest odd, binary sequence? Is 10 the smallest even, binary sequence? It turns out that the answer to the above questions is in the negative, which is clearly not encouraging.

A simple redefinition of \( B^0 \) and \( B^1 \) solves that problem. It also makes the specialized functions a lot elegant.

The new definitions are in Figure 2. The changes are in the terminating conditions of the abstract syntax and the rules. The terminal terms 10 and 1 have been removed from the definitions of \( B^0 \) and \( B^1 \) respectively. The term 0 now terminates the recursive definition of \( B^0 \). And the definitions for \( B^1 \) do not have a terminating value, they terminate using \( B^0 \). In the new definition the numeric value 1 is represented by the binary sequence: 0.01 and the number value 2 is represented by the binary sequence 0.10 (where the “.” is a punctuation symbol). Thus, additional zeros are prefixed to the binary sequence representing a positive number. There may actually be an arbitrarily large number of zeros prefixed, thereby having more than one binary sequence representing every numeric value. That does not effect the theoretic argument. The change of terminating condition also has the effect that the domain \( B^0 \) now no longer contains just even number, it also contains the sequence 0, 0.0, 0.0.0, and etc., all representing the numeric value 0. This only require changing the specializations of function \( h \). These functions now terminate on processing 0. This does not however effect the rules for the recursive cases.
3 Observations

Observe the symmetry in equations 6b-d and 7b-d. Replacing in equations 6b-d all occurrences of $f$ by $h$, $h$ by $f$, 1 by 0, 0 by 1, $k$ by $j$, and $j$ by $k$ gives equations 7b-d, respectively. The converse is also true. The symmetry does not hold for the equations for the base case (of the recursion). This symmetry is also reflected in the functions of Figure 1 created from specializing $f$ and $h$. The figure is organized to place the symmetric functions in the same column.

Let $\ell : B^+ \rightarrow N^+$ give the number of bits (i.e., 0 or 1) in an element in $B^+$. Let $\vartheta : B^+ \rightarrow N^+$ give the number in $N^+$ represented by the binary sequence.

To prove the $3x + 1$ Conjecture it is sufficient to show that for all $j \in B^1 - \{1\}$ there exists $r \in N^+$ such that $\vartheta(\tilde{T}^r(j1)) < \vartheta(j1)$. Alternatively, one may also show that there exists $r \in N^+$ such that $\ell(\tilde{T}^r(j1)) < \ell(j1)$. The length of the sequence gives an additional property that may be used in proving the conjecture.

**Lemma:** $\forall k, j. \ell(h(k0)) \leq \ell(k0) + 1$ and $\ell(f(j1)) \leq \ell(j1) + 1$.

**Proof:** From structural induction.

In other words, as a result of the mappings due to the functions $f$ and $g$ the length of a sequence is increased by at most one bit.

**Lemma:** $\forall j. \ell(\tilde{T}^r(j1)) \leq \ell(j1) + 1$. 
Lemma: For all $k_0, k'_0, k''_0 \in B^0$
\[ j_1, j'_1, j''_1 \in B^1 \]
\[ k_0 := k'_0.0 | k''_0.10 | j_1.10 | 0 \]
\[ j_1 := j'_1.1 | j''_1.01 | k_0.01 \]
\[ \tilde{T} : B^1 \rightarrow B^1 \]
\[ \tilde{T}^\prime(0.01) = 0.01 \]
\[ \tilde{T}^\prime(j_1.1) = f_1(j_1) \]
\[ \tilde{T}^\prime(j_1.01) = \tilde{T}^\prime(j_1) \]
\[ \tilde{T}^\prime(k_0.01) = h_1(k_0) \]

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<tr>
<th>$f_0 : B^1 \rightarrow B^0$</th>
<th>$f_1 : B^1 \rightarrow B^1$</th>
<th>$f_{10} : B^1 \rightarrow B^0$</th>
<th>$f_{01} : B^1 \rightarrow B^1$</th>
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<th>$h_0 : B^0 \rightarrow B^0$</th>
<th>$h_{01} : B^0 \rightarrow B^1$</th>
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<td>$h_1(j_1.10) = f_{01}(j_1).1$</td>
<td>$h_0(j_1.10) = f_0(j_1).10$</td>
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<td>$h_{10}(j_1.10) = f_0(j_1).10$</td>
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Figure 2 A revised version of the term-rewriting formulation from the previous Figure.

**Proof:** Follows from previous lemma.

The new formulation $\tilde{T}^\prime$, Figure 2, uses specialized versions of functions $f$ and $h$ with the following properties.

**Lemma:** $\ell(f_1(j_1)) \leq \ell(j_1) + 2$ and $\ell(h_1(k_0)) \leq \ell(k_0) + 2$

**Proof:** By structural induction.

Thus, only the second rule of $\tilde{T}^\prime$, Figure 2, maps a sequence to a longer sequence. The other rules either map to a smaller sequence or a sequence of the same length.

**Lemma:** $\vartheta(\tilde{T}(k_0.01)) \leq \vartheta(k_0.01)$

**Proof:** $\vartheta(\tilde{T}(k_0.01)) = \vartheta(h_1(k_0)) = \vartheta((k_0 \oplus k)1) = \vartheta(k_00 \oplus k0 \oplus 1) \leq \vartheta(k_00 \oplus k00 \oplus 1) = \vartheta(k_0.01)$.

So even though mappings due to the fourth rule of $\tilde{T}^\prime$, Figure 1, may not always decrease the length of a sequence, it does decrease its value.

The following assertions restate assertions proven by Andaloro [1]

**Corollary:** $\tilde{T}(j_1.(01)^n) = \tilde{T}(j_1)$.

**Lemma:** (Andaloro [1, Lemma 3]). For all $n \in N^+$:
\[ \tilde{T}^2(k_0.01.1^{2n-2}) = \tilde{T}^2(k_0.01.1^{2n-1}) \]
\[ \tilde{T}^2(j_1.01.1^{2n-1}) = \tilde{T}^2(j_1.01.1^{2n}). \]
4 Conclusions

I started this investigation with the following premise: “If the $3x + 1$ Conjecture is valid then there must exist a partial order $\subseteq$ over which $T(x) \subseteq x$. Find the partial order and you’ve proven the conjecture.” The ordering relation $\leq$ is based on the abstract representation of natural numbers as $0$, $s(0)$, $s(s(0))$, and etc. This relation and structure does not appear to be adequate to derive another partial order that may help in proving or disproving the Conjecture. The abstract syntax domain for even and odd numbers I have presented provide an alternative structure. I hope that this structure may provide insight into a partial ordering over which $T(x) \subseteq x$. I have investigated some partial orders based on structured containedness, but have not achieve success.

Bibliography


